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REPTATION MOTION OF LARGE ANIMALS IN A FLUID

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An asymptotic analysis of the plane problem of reptation motion of animals in a fluid is performed in a long-wave approximation. Turbulent motion is considered. Asymptotic estimates are obtained for the axial and shear forces, expended energy, and motion trajectory. Results of numerical analysis are given.

Key words: *turbulence, reptation motion, elastic line.*

In [1, 2], the problem in question was analyzed using the principle formulated by Lavrent'ev [3]. According to Lavrent'ev's principle, the animal's body is treated as an elastic rod placed in a solid channel of variable curvature. The environment surrounding the body plays the role of the solid walls of the channel. For motion in a fluid, the fluid plays the role of the channel walls because under rapid action for a time during which the organism moves a considerable distance, the fluid remains nearly motionless relative to the initial position by virtue of its inertia.

Kuznetsov et al. [2] considered irrotational motion in an ideal fluid, which is equivalent to motion in a freely moving solid channel whose mass depends on the shape. For transverse flow around a cylinder, the potential was determined by the method of plane sections. The axial friction force was ignored since in an ideal fluid the shear stress on the body surface is equal to zero.

Shapovalov [4] studied the laminar reptation motion of animals. The results obtained applies to the motion of microorganisms. The present paper extends the approach of [4] to the turbulent motion of animals. This is true for animals of large sizes, such as eel, moray, etc.

The problem of plane reptation motion of large animals in a fluid is formulated and solved in a long-wave approximation. The energy, force, and kinematic characteristics of the motion are determined. The results of numerical analysis are given.

1. Formulation of the Problem. We consider a developed turbulent regime that corresponds to the quadratic resistance law.

We study the motion of animals whose body is prolate enough (eels, water snakes, etc.) to satisfy the condition $l \gg d$ (l and d are the length of the body in the prolate state and its diameter). The elastic axis passes along the backbone. The backbone can be treated as a hinged system of rods. The number of vertebras is considered infinite, and the elastic axis is treated as a monotonic smooth curve.

The central nervous system sends command signals to the body muscles, so that a nearly sinusoidal traveling wave is formed. The number of muscles is considered infinite, and the command signal is a continuous monotonic function.

The Archimedean force is ignored since the density of the animal's body is close to the density of the surrounding fluid. The cross section of the body is constant along its length. If the surrounding fluid is conditionally considered motionless, the dissipation of mechanical energy is localized in a region commensurable with the cross-sectional dimensions of the animal, i.e., in the hydrodynamic boundary layer.

The longitudinal and transverse friction forces (dP and dF, respectively) act on an elementary segment of the body of length ds. Oblique flow around the cylinder is the case. For the longitudinal friction force, Zyabitskii [5], using the theory of a turbulent boundary layer for the case of a motionless cylinder, obtained the expression

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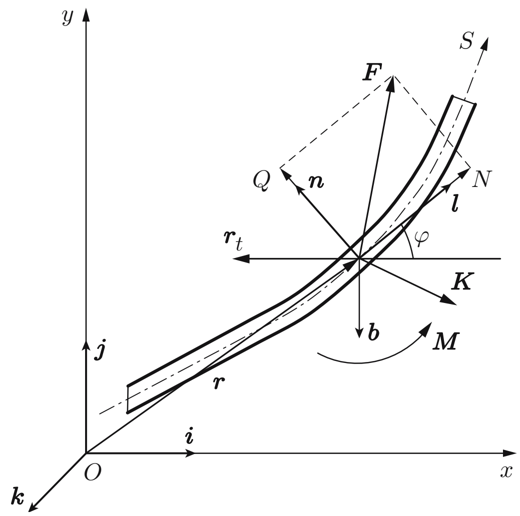

Fig. 1. Diagram of reptation motion.

$dP = 0.325\,\mathrm{Re}_l^{-0.7}\,\pi d\rho v_l^2\,ds$ (v_l is the axial velocity of motion of the cylinder is obtained, $\mathrm{Re}_l = v_l d\rho/\mu$ is the Reynolds number, and ρ and μ are the density and viscosity of the fluid, respectively), which is in good agreement with experimental data.

According to [6], the shear friction force is described by the expression $dF = 0.5\xi d\rho v_n^2\,ds$, where v_n is the normal velocity component of the cylinder, and ξ is the drag coefficient dependent on the Reynolds number $\mathrm{Re}_n = v_n d\rho/\mu$. For developed turbulent motion, Kochin et al. [7] obtained the similar relation $dF = 0.48 d\rho v_n^2\,ds$. Except in the critical regime, the drag coefficient depends weakly on the Reynolds number. Thus, in the range $\mathrm{Re}_n = 10$–10^4, the values of ξ decrease monotonically from 1.3 to 1. In the critical regime ($\mathrm{Re}_n = 5 \cdot 10^5$), we have $\xi = 0.3$ [6].

During motion, the transverse and longitudinal velocity components change under a periodic law. As a first approximation for the longitudinal and transverse components of the friction forces, we use the quadratic resistance law corresponding to the developed turbulent regime. Quite often, the body of water animals has an elliptic cross section, which improves its hydrodynamic properties. In this case, the friction force components differ only in constant coefficients. Let us consider a body that has a circular cross section which is constant along the length. The drag is ignored.

The friction force components are written as

$$dP = A_m v_l^2\,ds, \qquad dF = B_m v_n^2\,ds,$$

where $A_m = 0.325\,\mathrm{Re}_l^{-0.7}\,\pi d\rho$ and $B_m = 0.5\xi d\rho$. The parameters A_m and B_m have constant values.

During directional movement, the animal performs plane reptation motion, for example in the horizontal plane (Fig. 1). During this motion, the elastic axis and the acting forces lie in the plane xOy. Let us introduce a spatially stationary coordinate system (x, y, z), where x, y, and z are the coordinates of the points of the elastic line of the body s. The vector function $\boldsymbol{r}(s,t)$, $0 \leqslant s \leqslant l$ (t is time) performs vector parametrization of the curve s. The directions x, y, and z correspond to a right-hand oriented trihedron $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. We denote by $\boldsymbol{l}$ ($\boldsymbol{l} = \boldsymbol{r}_s$ and $|\boldsymbol{l}| = 1$) the tangent vector to the elastic line, $\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{l}$ is the normal vector, and $\boldsymbol{b}$ is the binormal vector.

The animal needs to overcome not only the resistance of the environment but also the inertia force of its own body. The density of the body is assumed to be equal to the density of the surrounding fluid ρ.

The equilibrium equations are written as

$$\boldsymbol{F}_s = -\boldsymbol{K}, \qquad \boldsymbol{M}_s + \boldsymbol{m} = \boldsymbol{F} \times \boldsymbol{l},$$

where $\boldsymbol{M}$ is the moment, $\boldsymbol{F} = (\boldsymbol{F} \cdot \boldsymbol{l})\,\boldsymbol{l} + (\boldsymbol{F} \cdot \boldsymbol{n})\,\boldsymbol{n} = N\boldsymbol{l} + Q\boldsymbol{n}$ is the force, $\boldsymbol{K}$ is the linear density of the external forces, including the inertia force, $\boldsymbol{m}$ is the distributed moment of the external load, N is the longitudinal force, and Q is the shear force; the subscripts denote the corresponding derivatives.

The expression for the external force vector taking into account the quadratic resistance law and the inertia force of the body (the surrounding fluid is motionless) is written as

$$\boldsymbol{K} = A_m|\boldsymbol{r}_t|\,\boldsymbol{l}\,(\boldsymbol{r}_t\cdot\boldsymbol{l}) + B_m|\boldsymbol{r}_t|\,\boldsymbol{n}\,(\boldsymbol{r}_t\cdot\boldsymbol{n}) - \rho(\pi d^2/4)\,\boldsymbol{l}\,(\boldsymbol{r}_{tt}\cdot\boldsymbol{l}) - \rho(\pi d^2/4)\,\boldsymbol{n}\,(\boldsymbol{r}_{tt}\cdot\boldsymbol{n}),$$

where $|\boldsymbol{r}_t| = \sqrt{x_t^2+y_t^2}$ is the velocity modulus of the elastic axis of the animal. The distributed moment of the external load $\boldsymbol{m}$ is due to the rotational moment of inertia of the cross section [8]: $\boldsymbol{m} = -\rho J\varphi_{tt}\boldsymbol{b}$, where $J = \pi d^4/64$ is the moment of inertia of the cross section of the body (which is constant along its length).

We have the following equations in scalar form:

$$N_s - Q\varphi_s = -A_m\sqrt{x_t^2+y_t^2}\,(x_t\cos\varphi + y_t\sin\varphi) + \rho(\pi d^2/4)(x_{tt}\cos\varphi + y_{tt}\sin\varphi),$$

$$N\varphi_s + Q_s = -B_m\sqrt{x_t^2+y_t^2}\,(-x_t\sin\varphi + y_t\cos\varphi) + \rho(\pi d^2/4)(-x_{tt}\sin\varphi + y_{tt}\cos\varphi), \tag{1.1}$$

$$M_s - \rho J\varphi_{tt} = -Q.$$

These equations are written with allowance for the relations $\boldsymbol{r}_t\cdot\boldsymbol{l} = x_t\cos\varphi + y_t\sin\varphi$, $\boldsymbol{r}_t\cdot\boldsymbol{n} = -x_t\sin\varphi + y_t\cos\varphi$, $\boldsymbol{r}_{tt} = x_{tt}\boldsymbol{i} + y_{tt}\boldsymbol{j}$, and $\boldsymbol{M} = M\boldsymbol{b}$.

We have a system with distributed parameters. According to the last equation in (1.1), the muscles located symmetrically about the backbone of the animal produce a moment, which is expended in overcoming the inertia forces due to the rotation of the body cross section and in producing a shear force. In turn, the shear force is expended in overcoming the inertia forces due to the transverse motion of the body and the hydrodynamic resistance of the surrounding medium.

We transform to dimensionless parameters and variables using the largest value of the shear force Q ($Q_0 = |\max Q|$) as the force scale:

$$\{X,Y,S\} = \{x,y,s\}l^{-1},\quad e = \frac{A_m}{B_m},\quad n = \frac{N}{Q_0},\quad q = \frac{Q}{Q_0},\quad \tau = t\sqrt{\frac{Q_0}{A_m l^3}},$$

$$\mathrm{In} = \frac{\rho\pi d^2}{4A_m l},\quad \Omega = \omega\sqrt{\frac{A_m l^3}{Q_0}},\quad K = kl,\quad w = W\sqrt{\frac{A_m l}{Q_0^3}}\,.$$

Here ω is the frequency of muscle contraction, w is the expended energy, and In is the inertia parameter.

In dimensionless form, Eqs. (1.1) supplemented by geometrical relations and boundary conditions have the form

$$n_s - q\varphi_s = f_1,\qquad f_1 = -\sqrt{X_\tau^2+Y_\tau^2}\,(X_\tau\cos\varphi + Y_\tau\sin\varphi) + \mathrm{In}\,(X_{\tau\tau}\cos\varphi + Y_{\tau\tau}\sin\varphi),$$

$$n\varphi_s + q_s = f_2,\qquad f_2 = -e^{-1}\sqrt{X_\tau^2+Y_\tau^2}\,(-X_\tau\sin\varphi + Y_\tau\cos\varphi) + \mathrm{In}\,(-X_{\tau\tau}\sin\varphi + Y_{\tau\tau}\cos\varphi),$$

$$X_s = \cos\varphi,\qquad Y_s = \sin\varphi, \tag{1.2}$$

$$\tau = 0:\qquad X = X^0(S),\quad Y = Y^0(S),$$

$$\tau > 0,\ S = 0:\quad n = q = 0,\qquad S = 1:\quad n = q = 0.$$

The functions X^0 and Y^0 describe the initial configuration.

The nervous impulses transmitted to the animal's muscles form a traveling wave, which ensures translational motion. In Eqs. (1.2), it is necessary to specify one of the functions n, q, and φ *a priori*. For the plane traveling wave, we use the expression

$$\varphi = \varepsilon\sin(KS - \Omega\tau), \tag{1.3}$$

where Ω is the dimensionless frequency of muscle contraction, ε is a dimensionless parameter ($|\varepsilon| \leqslant \pi/2$), and $K = 2\pi i$ ($i = 1,2,3\ldots$). According to the last equation, the body length is a multiple of an integer of waves, which considerably simplifies the computational expressions.

For $d = 0.05$ m, $v = 1$ m/sec, $\rho = 10^3$ kg/m^3, and $\mu = 10^{-3}$ Pa·sec, the inertia parameter is In = 74.86. Therefore, it is necessary to take into account the last terms on the right side of the first two equations (1.2).

In Eqs. (1.2), the factors do not contain the moment of inertia of the cross section. Therefore, the inertia forces due to the rotational inertia of the cross sections do not influence the motion trajectory of the animal's body and only determine the moment (1.1).

According to [4], the energy W can be defined by the integral

$$W = \int_0^l \boldsymbol{r}_t \cdot \boldsymbol{K}\, ds.$$

In view of the relation

$$N_s - Q\varphi_s = -A_m|\boldsymbol{r}_t|(\boldsymbol{r}_t \cdot \boldsymbol{l}) + \rho\,\frac{\pi d^2}{4}\,(\boldsymbol{r}_{tt} \cdot \boldsymbol{l}), \quad N\varphi_s + Q_s = -B_m|\boldsymbol{r}_t|(\boldsymbol{r}_t \cdot \boldsymbol{n}) + \rho\,\frac{\pi d^2}{4}\,(\boldsymbol{r}_{tt} \cdot \boldsymbol{n}),$$

$$\boldsymbol{r}_t = x_t\boldsymbol{i} + y_t\boldsymbol{j}$$

and Eqs. (1.2), the expression for the dimensionless power becomes

$$w = \int_0^1 \Big\{ \sqrt{X_\tau^2 + Y_\tau^2}\, [(x_t \cos\varphi + y_t \sin\varphi)^2 + e^{-1}(-x_t \sin\varphi + y_t \cos\varphi)^2] \Big\}\, dS. \tag{1.4}$$

2. Solution of the Problem. We assume that the axial load and shear force are functions of φ, i.e., $n = n(\varphi)$ and $q = q(\varphi)$, where $\varphi = \varphi(S, \tau)$. In this case, the first two equations in (1.2) become

$$n_\varphi - q = f_1\varphi_s^{-1}, \qquad n + q_\varphi = f_2\varphi_s^{-1}. \tag{2.1}$$

Eliminating the function q from these equations, we obtain the following inhomogeneous linear equation of the second order for the function n:

$$n_{\varphi\varphi} + n = f_2\varphi_s^{-1} + (f_1\varphi_s^{-1})_\varphi.$$

The solution has the form

$$n = C_1 \sin\varphi + C_2 \cos\varphi - \cos\varphi \int [f_2\varphi_s^{-1} + (f_1\varphi_s^{-1})_\varphi] \sin\varphi\, d\varphi + \sin\varphi \int [f_2\varphi_s^{-1} + (f_1\varphi_s^{-1})_\varphi] \cos\varphi\, d\varphi, \tag{2.2}$$

where C_1 and C_2 are constants.

Integrating the integrands by parts, we obtain the equalities

$$\int (f_1\varphi_s^{-1})\varphi \sin\varphi\, d\varphi = f_1\varphi_s^{-1} \sin\varphi - \int f_1 \cos\varphi\, dS,$$

$$\int (f_1\varphi_s^{-1})_\varphi \cos\varphi\, d\varphi = f_1\varphi_s^{-1} \cos\varphi + \int f_1 \sin\varphi\, dS.$$

In view of these relations, expression (2.2) becomes

$$n = C_1 \sin\varphi + C_2 \cos\varphi - \cos\varphi \Big(\int_0^S f_2 \sin\varphi\, dS - \int_0^S f_1 \cos\varphi\, dS \Big) + \sin\varphi \Big(\int_0^S f_2 \cos\varphi\, dS + \int_0^S f_1 \sin\varphi\, dS \Big). \tag{2.3}$$

According to the first equation in (1.2), the shear force is $q = \varphi_s^{-1}(n_s - f_1)$. Taking into account expressions (2.3), we have

$$q = C_1 \cos\varphi - C_2 \sin\varphi + \sin\varphi \Big(\int_0^S f_2 \sin\varphi\, dS - \int_0^S f_1 \cos\varphi\, dS \Big) + \cos\varphi \Big(\int_0^S f_2 \cos\varphi\, dS + \int_0^S f_1 \sin\varphi\, dS \Big). \tag{2.4}$$

The constants in expressions (2.3) and (2.4) are found from the condition of no forces at the left end ($S = 0$ and $n = q = 0$) of the body (1.2). Thus, we obtain the system of equations

$$C_1 \sin\varphi_0 + C_2 \cos\varphi_0 = 0, \qquad C_1 \cos\varphi_0 - C_2 \sin\varphi_0 = 0,$$

where $\varphi_0(\tau) = \varphi|_{S=0}$. The solution of this system has the form $C_1 = C_2 = 0$.

Taking into account the boundary conditions for the right end ($S = 1$ and $n = q = 0$) of the animal's body (1.2), we have the system of equations

$$-\cos\varphi_0\Big(\int\limits_0^1 f_2\sin\varphi\,dS-\int\limits_0^1 f_1\cos\varphi\,dS\Big)+\sin\varphi_0\Big(\int\limits_0^1 f_2\cos\varphi\,dS+\int\limits_0^1 f_1\sin\varphi\,dS\Big)=0,$$

$$\sin\varphi_0\Big(\int\limits_0^1 f_2\sin\varphi\,dS-\int\limits_0^1 f_1\cos\varphi\,dS\Big)+\cos\varphi_0\Big(\int\limits_0^1 f_2\cos\varphi\,dS+\int\limits_0^1 f_1\sin\varphi\,dS\Big)=0.$$

Here we took into account the property of function (1.3) $\varphi_0 = \varphi|_{S=0} = \varphi|_{S=1}$. The system of trigonometric equations has a trivial solution:

$$\int\limits_0^1(f_2\sin\varphi-f_1\cos\varphi)\,dS=0,\qquad\int\limits_0^1(f_2\cos\varphi+f_1\sin\varphi)\,dS=0. \tag{2.5}$$

Substituting the expressions for the functions f_1 and f_2 from (1.2) into (2.5), we write the equations in expanded form

$$\int\limits_0^1\Big(\sqrt{X_\tau^2+Y_\tau^2}\,\Big\{X_\tau[1+(e^{-1}-1)\sin^2\varphi]+0.5(1-e^{-1})Y_\tau\sin 2\varphi\Big\}-\mathrm{In}\,X_{\tau\tau}\Big)\,dS=0,$$

$$\int\limits_0^1\Big(\sqrt{X_\tau^2+Y_\tau^2}\,\Big\{0.5(e^{-1}-1)X_\tau\sin 2\varphi-Y_\tau[e^{-1}+(1-e^{-1})\sin^2\varphi]\Big\}+\mathrm{In}\,Y_{\tau\tau}\Big)\,dS=0. \tag{2.6}$$

The geometrical relation for the elastic axis from (1.2) and expression (1.3) leads to the following equations for the functions X and Y:

$$X_S=\cos\varphi=1-\varphi^2/2!+\ldots=1-(\varepsilon^2/2)\sin^2(KS-\Omega\tau)+\ldots\,,$$

$$Y_S=\sin\varphi=\varphi-\varphi^3/3!+\ldots=\varepsilon\sin(KS-\Omega\tau)-(\varepsilon^3/6)\sin^3(KS-\Omega\tau)+\ldots\,. \tag{2.7}$$

We will analyze the problem using the small parameter method with the geometrical perturbation amplitude ε in (1.3) as the small parameter. The functions X and Y are obtained in the form of direct expansions in the powers of the small parameter. According to expansions (2.7), the required functions can be written as

$$X=X_0+\varepsilon^2X_2+\ldots,\qquad Y=\varepsilon Y_1+\varepsilon^3Y_3^3+\ldots,\qquad|\varepsilon|\ll 1. \tag{2.8}$$

A detailed analysis of the problem shows that terms with odd powers in ε in X and with even powers in Y are equal to zero. Retaining the first two terms of the expansion for the function X and one term for Y and integrating Eqs. (2.7) from 0 to S with allowance for (2.8), we obtain

$$X_0=S+C_3(\tau),\qquad X_2=-\frac{1}{2K}\Big(\frac{KS}{2}-\frac{\sin 2(KS-\Omega\tau)}{4}-\frac{\sin 2\Omega\tau}{4}\Big)+C_4(\tau),$$

$$Y_1=-(\cos(KS-\Omega\tau)-\cos\Omega\tau)/K+C_5(\tau), \tag{2.9}$$

where C_3, C_4, and C_5 are unknown functions of time.

The function C_3 characterizes the motion of the animal along the X axis. However, since the component X_0 does not depend on the perturbation amplitude ε, it is necessary to set $C_3 = 0$.

At the initial time, let the left end of the animal's body be at the cross section $X = 0$ and the elastic axis be symmetric about the X axis (the static moment of the elastic axis with respect to the X axis is equal to zero). Thus, we have the conditions

$$\tau=0,\quad S=0:\qquad X=0; \tag{2.10}$$

$$\tau = 0: \qquad \int_0^1 Y\,dS = 0. \tag{2.11}$$

Taking into account relations (2.8) and (2.9) and condition (2.10), we obtain the equality $\varepsilon^2 C_4|_{\tau=0} + \ldots = 0$, which leads to the following initial condition for the unknown function of time:

$$\tau = 0, \qquad C_4 = 0. \tag{2.12}$$

Substituting (2.8) and (2.9) into (2.11), for any time we have

$$\varepsilon(K^{-1}\cos\Omega\tau + C_5) + \ldots = 0.$$

Here we took into account the equalities $K = 2\pi n$ ($n = 1, 2, 3\ldots$; $\sin K = 0$ and $\cos K = 1$). Thus, the function C_5 has the form

$$C_5 = -K^{-1}\cos\Omega\tau. \tag{2.13}$$

According to (2.8), (2.9), and (2.13), the function Y we write as

$$Y = -(\varepsilon/K)\cos(KS - \Omega\tau) + O(\varepsilon^3). \tag{2.14}$$

To find the function $C_4(\tau)$, we use Eqs. (2.6). Substitution of the expansions (2.8) into (2.6) (only terms of the orders considered are taken into account) yields

$$\int_0^1 \Big[\Big(\varepsilon|Y_{1\tau}| + \frac{\varepsilon^3}{2}\frac{X_{2\tau}^2}{Y_{1\tau}}\Big)\Big\{\varepsilon^2 X_{2\tau}[1 + (e^{-1} - 1)(\varepsilon\varphi_1)^2] + (1 - e^{-1})\varepsilon^2 Y_{1\tau}\varphi_1\Big\} - \mathrm{In}\,\varepsilon^2 X_{2\tau\tau}\Big]\,dS = 0,$$

$$\int_0^1 \Big[\Big(\varepsilon|Y_{1\tau}| + \frac{\varepsilon^3}{2}\frac{X_{2\tau}^2}{Y_{1\tau}}\Big)\Big\{(e^{-1} - 1)\varepsilon^3 X_{2\tau}\varphi_1 - \varepsilon Y_{1\tau}[e^{-1} + (1 - e^{-1})\varepsilon^2\varphi_1^2]\Big\} + \mathrm{In}\,\varepsilon Y_{1\tau\tau}\Big]\,dS = 0.$$

Here $\varphi_1 = \sin(KS - \Omega\tau)$ and we used the relations

$$\sin^2\varphi \approx \varepsilon^2\varphi_1^2 + \ldots, \quad \sin 2\varphi \approx 2\varepsilon\varphi_1 + \ldots, \quad \sqrt{(\varepsilon^2 X_{2\tau})^2 + (\varepsilon Y_{1\tau})^2} \approx \varepsilon|Y_{1\tau}| + \frac{\varepsilon^3}{2}\frac{X_{2\tau}^2}{Y_{1\tau}} + \ldots\,.$$

Collecting the coefficients of the same powers of ε, we obtain the equations

$$\int_0^1 Y_{1\tau\tau}\,dS = 0 \quad \text{for } \varepsilon^1; \tag{2.15}$$

$$\int_0^1 X_{2\tau\tau}\,dS = 0, \qquad \int_0^1 Y_{1\tau}^2\,dS = 0 \quad \text{for } \varepsilon^2; \tag{2.16}$$

$$\int_0^1 \Big[|Y_{1\tau}|X_{2\tau} + |Y_{1\tau}|(1 - e^{-1})Y_{1\tau}\varphi_1\Big]\,dS = 0 \quad \text{for } \varepsilon^3. \tag{2.17}$$

Equation (2.15) and the second equation in (2.16) are identities. From the first equation of (2.16) using (2.9), we find the function C_4:

$$C_4 = -\frac{1}{8K}\sin 2\Omega\tau + C_{40}\tau + C_{41} \tag{2.18}$$

(C_{40} and C_{41} are constants). Condition (2.12) implies that $C_{41} = 0$. The constant C_{40} is found using Eq. (2.17). Integration of this equation with allowance for expressions (2.9), (2.14), and (2.18) yields

$$C_{40} = -\frac{\Omega}{12K}\,\frac{8 - 7e}{e}.$$

Taking into account relations (2.8), (2.9), and (2.18), for the function X we rite the expression

$$X = S\Big(1 - \frac{\varepsilon^2}{4}\Big) + \frac{\varepsilon^2}{8K}\sin 2(KS - \Omega\tau) - \frac{\varepsilon^2\Omega(8 - 7e)\tau}{12Ke} + O(\varepsilon^4). \tag{2.19}$$

After simple transformations, expressions (2.3) and (2.4) become

$$n = -\cos\varphi \int_0^S \Big(\sqrt{X_\tau^2 + Y_\tau^2}\,\Big\{X_\tau[1 + (e^{-1} - 1)\sin^2\varphi] + 0.5(1 - e^{-1})Y_\tau \sin 2\varphi\Big\} - \operatorname{In} X_{\tau\tau}\Big)\, dS$$

$$+ \sin\varphi \int_0^S \Big(\sqrt{X_\tau^2 + Y_\tau^2}\,\Big\{0.5(e^{-1} - 1)X_\tau \sin 2\varphi - Y_\tau[e^{-1} + (1 - e^{-1})\sin^2\varphi]\Big\} + \operatorname{In} Y_{\tau\tau}\Big)\, dS,$$

$$q = \sin\varphi \int_0^S \Big(\sqrt{X_\tau^2 + Y_\tau^2}\,\Big\{X_\tau[1 + (e^{-1} - 1)\sin^2\varphi] + 0.5(1 - e^{-1})Y_\tau \sin 2\varphi\Big\} - \operatorname{In} X_{\tau\tau}\Big)\, dS$$

$$+ \cos\varphi \int_0^S \Big(\sqrt{X_\tau^2 + Y_\tau^2}\,\Big\{0.5(e^{-1} - 1)X_\tau \sin 2\varphi - Y_\tau[e^{-1} + (1 - e^{-1})\sin^2\varphi]\Big\} + \operatorname{In} Y_{\tau\tau}\Big)\, dS.$$

Substitution of formulas (2.14) and (2.19) into these expressions yields the following asymptotic estimates for the axial force and the shear force:

$$\begin{aligned}
n &= \varepsilon^2 \frac{\Omega^2}{K^2} \cos[\varepsilon \sin(KS - \Omega\tau)]\Big[\frac{2-e}{3e} J_1 + \Big(\frac{1}{2} - e^{-1}\Big) J_3\Big] + \varepsilon^2 \frac{\Omega^2}{eK^2} \sin[\varepsilon\sin(KS - \Omega\tau)] J_2 \\
&\quad - \varepsilon^2 \frac{\Omega^2}{4K^2} \operatorname{In}[\cos 2(KS - \Omega\tau) - \cos 2\Omega\tau] \cos[\varepsilon \sin(KS - \Omega\tau)] \\
&\quad + \varepsilon^2 \frac{\Omega^2}{eK^2} \operatorname{In}[\sin(KS - \Omega\tau) - \sin\Omega\tau] \sin[\varepsilon\sin(KS - \Omega\tau)] + O(\varepsilon^4), \\
q &= -\varepsilon^2 \frac{\Omega^2}{K^2} \sin[\varepsilon \sin(KS - \Omega\tau)]\Big[\frac{2-e}{3e} J_1 + \Big(\frac{1}{2} - e^{-1}\Big) J_3\Big] + \varepsilon^2 \frac{\Omega^2}{eK^2} \cos[\varepsilon\sin(KS - \Omega\tau)] J_2 \\
&\quad + \varepsilon^2 \frac{\Omega^2}{4K^2} \operatorname{In}[\cos 2(KS - \Omega\tau) - \cos 2\Omega\tau] \sin[\varepsilon \sin(KS - \Omega\tau)] \\
&\quad + \varepsilon^2 \frac{\Omega^2}{eK^2} \operatorname{In}[\sin(KS - \Omega\tau) - \sin\Omega\tau] \cos[\varepsilon\sin(KS - \Omega\tau)] + O(\varepsilon^4).
\end{aligned} \tag{2.20}$$

Here

$$J_1 = \int_0^S |\sin(KS - \Omega\tau)|\, dS = \frac{2}{\pi K}\Big(KS - 2\sum_{n=1}^{\infty} \frac{1}{2n(4n^2-1)} \{\sin[2n(KS - \Omega\tau)] + \sin(2n\Omega\tau)\}\Big),$$

$$J_2 = \int_0^S |\sin(KS - \Omega\tau)| \sin(KS - \Omega\tau)\, dS$$

$$= \frac{1}{2K}\Big\{|\sin(KS - \Omega\tau)|\cos(KS - \Omega\tau) - |\sin(\Omega\tau)|\cos(\Omega\tau) + \arcsin[\cos(KS - \Omega\tau)] - \arcsin[\cos(\Omega\tau)]\Big\},$$

$$J_3 = \int_0^S |\sin(KS - \Omega\tau)| \sin^2(KS - \Omega\tau)\, dS = \frac{1}{2} J_1 - \frac{1}{2\pi K}\Big(-\frac{2}{3} KS + \sin[2(KS - \Omega\tau)] + \sin(2\Omega\tau)$$

$$- \sum_{n=1}^{\infty} \frac{1}{(n+1)(4n^2-1)} \{\sin[2(n+1)(KS - \Omega\tau)] + \sin[2(n+1)\Omega\tau]\}$$

$$- \sum_{n=2}^{\infty} \frac{1}{(n-1)(4n^2-1)} \{\sin[2(n-1)(KS - \Omega\tau)] + \sin[2(n-1)\Omega\tau]\}\Big).$$

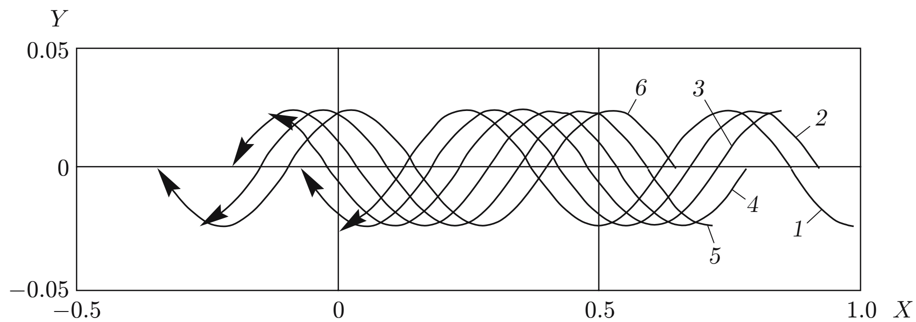

Fig. 2. Elastic force at times $\tau = 0$ (1), 0.25 (2), 0.5 (3), 0.75 (4), 1 (5), and 1.25 (6).

Using (2.14) and (2.19), we obtain the following asymptotic estimate for the expended power (1.4):

$$w = \varepsilon^3 \frac{\Omega^3}{eK^3} \int\limits_0^1 |\sin(KS - \Omega\tau)| \sin^2(KS - \Omega\tau)\, dS + O(\varepsilon^5).$$

Integration yields

$$w = 4\Omega^3\varepsilon^3/(3\pi e K^3) + O(\varepsilon^5).$$

The computational formula for the energy in dimensional form is written as

$$W = 2\omega^3 y_m^3 l\xi\rho d/(3\pi) + O(\varepsilon^5), \tag{2.21}$$

where $y_m = \varepsilon/k$ is the dimensional amplitude of the deflection of the elastic axis of the body from the x axis.

3. Analysis of the Solution. The first term on the right side of expression (2.19) in aggregate with expression (2.14) describes the elastic axis of the animal's body, and the second term describes the axial oscillations of the body during motion. The multiplicand of the third term $\Omega\varepsilon^2(8 - 7e)/(12Ke)$ characterizes the average velocity of motion of the animal along the X axis. The velocity does not depend on the inertia of the body (on the parameter In). For the law of motion (the function φ) specified *a priori*, the inertia forces influence the axial load and the shear force. The dimensional velocity $\bar{v}_x$ is defined by the expression $\bar{v}_x = -(8 - 7e)y_m^2 k\omega/(12e)$. The velocity depends largely on the relation between the friction force component (the parameter e). The result confirms Lavrent'ev's idea on the necessity of taking into account the viscous properties of the fluid [9].

Figure 2 gives the configurations of the elastic axis at various times. The calculations were performed using formulas (2.14) and (2.19) for $\varepsilon = 0.3$, $\Omega = 2\pi$, $K = 4\pi$, and $e = 0.1$. The animal moves to the left ($\Omega > 0$ and $K > 0$) along the X-axis. On each line of the elastic axis, the arrow shows the head of the animal.

The results of the numerical analysis of Eqs. (2.14), (2.19) suggest that to ensure the proper direction of the motion, the condition $e < 8/7$ should be satisfied. The parameter e characterizes the relation between the longitudinal and transverse friction forces and is defined by the formula $e = 0.65\pi\,\mathrm{Re}_l^{-0.7}$. The Reynolds number should satisfy the condition $\mathrm{Re}_l^{0.7} > 0.568\pi$.

As the longitudinal friction (the parameter e) decreases, the velocity increases; therefore, the velocity is greatly affected by the hydrodynamic boundary layer.

The axial velocity of an animal moving without friction in a glass tube of shape given by $Y = -(\varepsilon/K)\cos KS$ and $X = S(1 - \varepsilon^2/4)$ will have the maximum possible value equal to the traveling wave velocity Ω/K. The real velocity is lower. The parameters ε and e should satisfy the ultimate velocity condition $\varepsilon^2(8 - 7e)/(12e) \leqslant 1$.

Expressions (2.20) imply that the axial force and the shear force are cyclic in nature and proportional to the complex $\varepsilon^2\Omega^2/K^2$. The component due to the inertia forces is proportional to $\mathrm{In}\,\varepsilon^2\Omega^2/K^2$.

For turbulent motion, the expended power (2.21) differs significantly from the power for laminar motion [4]. As a first approximation, the inertia forces do not influence the expended power. The frequency of muscle contraction ω can be expressed in terms of the average velocity of motion: $\omega = |-12e\bar{v}_x/[(8 - 7e)y_m^2 k]|$.

The governing equation (1.3) does not always fit the real motion pattern. For example, during motion water snakes keep the axial orientation of the head. In addition, the span of the lateral vibrations increases from head to tail. In this case, it is possible to use the governing equation $\varphi = \varepsilon[\exp(aS) - 1]\sin(KS - \Omega\tau)$, where a is a constant.

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